Problem 1. Let $n \geqslant 100$ be an integer. The numbers $n, n+1, \ldots, 2n$ are written on $n+1$ cards, one number per card. The cards are shuffled and divided into two piles. Prove that one of the piles contains two cards such that the sum of their numbers is a perfect square.

Solution. To solve the problem it suffices to find three squares and three cards with numbers a, b, c on them such that pairwise sums $a+b, b+c, a+c$ are equal to the chosen squares. By choosing the three consecutive squares $(2k-1)^2$, $(2k)^2$, $(2k+1)^2$ we arrive at the triple

$$(a, b, c) = \left(2k^2 - 4k, \quad 2k^2 + 1, \quad 2k^2 + 4k\right).$$

We need a value for k such that

$$n \leqslant 2k^2 - 4k, \quad \text{and} \quad 2k^2 + 4k \leqslant 2n.$$

A concrete k is suitable for all n with

$$n \in \left[k^2 + 2k, 2k^2 - 4k + 1\right] =: I_k.$$

For $k \geqslant 9$ the intervals I_k and I_{k+1} overlap because

$$(k+1)^2 + 2(k+1) \leqslant 2k^2 - 4k + 1.$$

Hence $I_9 \cup I_{10} \cup \ldots = [99, \infty)$, which proves the statement for $n \geqslant 99$.

Comment 1. There exist approaches which only work for sufficiently large n.

One possible approach is to consider three cards with numbers $70k^2, 99k^2, 126k^2$ on them. Then their pairwise sums are perfect squares and so it suffices to find k such that $70k^2 \geqslant n$ and $126k^2 \leqslant 2n$ which exists for sufficiently large n.

Another approach is to prove, arguing by contradiction, that a and $a-2$ are in the same pile provided that n is large enough and a is sufficiently close to n. For that purpose, note that every pair of neighbouring numbers in the sequence $a, x^2-a, a+(2x+1), x^2+2x+3-a, a-2$ adds up to a perfect square for any x; so by choosing $x = \lfloor\sqrt{2a}\rfloor + 1$ and assuming that n is large enough we conclude that a and $a-2$ are in the same pile for any $a \in [n+2, 3n/2]$. This gives a contradiction since it is easy to find two numbers from $[n+2, 3n/2]$ of the same parity which sum to a square.

It then remains to separately cover the cases of small n which appears to be quite technical.

Comment 2. An alternative formulation for this problem could ask for a proof of the statement for all $n > 10^6$. An advantage of this formulation is that some solutions, e.g. those mentioned in Comment 1 need not contain a technical part which deals with the cases of small n. However, the original formulation seems to be better because the bound it gives for n is almost sharp, see the next comment for details.

Comment 3. The statement of the problem is false for $n = 98$. As a counterexample, the first pile may contain the even numbers from 98 to 126, the odd numbers from 129 to 161, and the even numbers from 162 to 196.

Problem 2: Show that for all real numbers $x_1, \ldots, x_n$ the following inequality holds:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \le \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}.$$

Solution A: If we add t to all the variables then the left-hand side remains constant and the right-hand side becomes

$$H(t) := \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j + 2t|}.$$

Let T be large enough such that both $H(-T)$ and $H(T)$ are larger than the value L of the left-hand side of the inequality we want to prove. Not necessarily distinct points $p_{i,j} := -(x_i + x_j)/2$ together with T and $-T$ split the real line into segments and two rays such that on each of these segments and rays the function $H(t)$ is concave since $f(t) := \sqrt{|\ell + 2t|}$ is concave on both intervals $(-\infty, -\ell/2]$ and $[-\ell/2, +\infty)$. Let $[a, b]$ be the segment containing zero. Then concavity implies $H(0) \ge \min\{H(a), H(b)\}$ and, since $H(\pm T) > L$, it suffices to prove the inequalities $H(-(x_i + x_j)/2) \ge L$, that is to prove the original inequality in the case when all numbers are shifted in such a way that two variables x_i and x_j add up to zero. In the following we denote the shifted variables still by x_i.

If $i = j$, i.e. $x_i = 0$ for some index i, then we can remove x_i which will decrease both sides by $2\sum_k \sqrt{|x_k|}$. Similarly, if $x_i + x_j = 0$ for distinct i and j we can remove both x_i and x_j which decreases both sides by

$$2\sqrt{2|x_i|} + 2 \cdot \sum_{k \ne i,j}\left(\sqrt{|x_k + x_i|} + \sqrt{|x_k + x_j|}\right).$$

In either case we reduced our inequality to the case of smaller n. It remains to note that for $n = 0$ and $n = 1$ the inequality is trivial.

Solution B: Consider an n by n matrix A with entries given by $a_{i,j} := \sqrt{|x_i + x_j|} - \sqrt{|x_i - x_j|}$. What we need to show is that

$$\vec{e}^{\,T} A \vec{e} \ge 0$$

for a vector $\vec{e}$ with all entries equal to 1. We will instead prove this inequality for all vectors $\vec{e} \in \mathbb{R}^n$, i.e. show that the matrix A is positive semidefinite.

For that it suffices to find a pre-Hilbert space V and a map $f : \mathbb{R} \to V$ such that $a_{i,j} = \langle f(x_i), f(x_j)\rangle$. In fact, this is equivalent to A being positive semidefinite. One possible choice of the space V and a map f is

$$V := L_2(\mathbb{R}_+), \qquad f(x) = c \cdot \frac{\sin(xt)}{t^{3/4}} \in L_2(\mathbb{R}_+),$$

for a specific choice of positive constant c. To prove the equality $a_{i,j} = \langle f(x_i), f(x_j)\rangle$, first, for a real p, consider the integral

$$I(p) = \int_0^\infty \frac{1-\cos(px)}{x\sqrt{x}}\,dx,$$

which clearly converges to a strictly positive number. By changing the variable $y = |p|x$ one notices that $I(p) = \sqrt{|p|}I(1)$. Hence, by using the trigonometric formula $\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$ we obtain

$$\sqrt{|a+b|} - \sqrt{|a-b|} = \frac{1}{I(1)}\int_0^\infty \frac{\cos((a-b)x) - \cos((a+b)x)}{x\sqrt{x}}\,dx = \frac{1}{I(1)}\int_0^\infty \frac{2\sin(ax)\sin(bx)}{x\sqrt{x}}\,dx,$$

which exactly gives us $a_{i,j} = \langle f(x_i), f(x_j)\rangle$ with the choice $c = \sqrt{2/I(1)}$.

Solution C: As in the previous solution, we want to show that A is positive semidefinite. Note that the property of A being positive semidefinite is invariant under the multiplication of any variable x_i by -1. So we assume that all x_i are non-negative.

For each real t consider the matrix $A^{(t)}$ with entries given by

$$a_{i,j}^{(t)} := \sqrt{|x_i + x_j + 2t|} - \sqrt{|x_i - x_j|},$$

so that $A = A^{(0)}$. We will show that $A^{(t)}$ positive semidefinite for all $t \geq -\min_i x_i$. For $t = -\min_i x_i$ all entries of one row and corresponding column vanish and so the problem reduces to the same problem for smaller n. Thus it remains to show that $dA^{(t)}/dt$ given by

$$\left(\frac{d}{dt}A^{(t)}\right)_{i,j} = \frac{1}{2\sqrt{x_i + x_j + 2t}}.$$

is positive semidefinite for $t > -\min_i x_i$. So the problem has been reduced to the fact that an n by n matrix B with entries

$$b_{i,j} = \frac{1}{\sqrt{y_i + y_j}}$$

is positive semidefinite for any tuple of positive real numbers $(y_1, \ldots, y_n)$. To prove that we use the same idea as in Solution B, we write

$$\frac{1}{\sqrt{a+b}} = c \cdot \int_0^\infty e^{-ta}e^{-tb}t^{-1/2}\,dt,$$

and so $1/\sqrt{y_i + y_j} = \langle f(y_i), f(y_j)\rangle$ for $f : \mathbb{R} \to V$, where V and f are given by

$$V = L_2(\mathbb{R}_+), \qquad f(x) = \sqrt{c}\cdot e^{-tx}t^{-1/4}.$$

Comment 1: A more general inequality

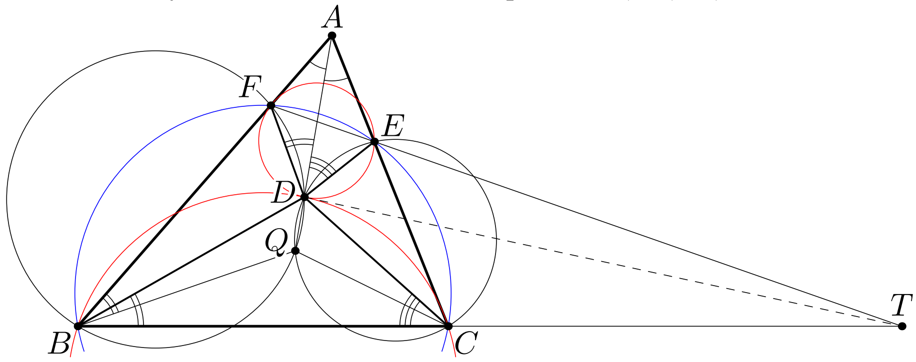
$$\sum_{i=1}^n \sum_{j=1}^n |x_i - x_j|^r \leq \sum_{i=1}^n \sum_{j=1}^n |x_i + x_j|^r$$

holds for any $r \in [0, 2]$. Solution A can be repeated verbatim for any $r \in [0, 1]$ but not for $r > 1$. In Solution B, by putting x^{r+1} in the denominator in place of $x\sqrt{x}$ we can prove the inequality for any $r \in (0, 2)$ and the cases $r = 0, 2$ are easy to check by hand.

Comment 2: In fact, the integral from Solution B can be computed explicitly, we have $I(1) = \sqrt{2\pi}$.

Problem 3. A point D is chosen inside an acute-angled triangle ABC with $AB > AC$ so that $\angle BAD = \angle DAC$. A point E is constructed on the segment AC so that $\angle ADE = \angle DCB$. Similarly, a point F is constructed on the segment AB so that $\angle ADF = \angle DBC$. A point X is chosen on the line AC so that $CX = BX$. Let O_1 and O_2 be the circumcentres of the triangles ADC and DXE. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Common remarks. Let Q be the isogonal conjugate of D with respect to the triangle ABC. Since $\angle BAD = \angle DAC$, the point Q lies on AD. Then $\angle QBA = \angle DBC = \angle FDA$, so the points Q, D, F, and B are concyclic. Analogously, the points Q, D, E, and C are concyclic. Thus $AF \cdot AB = AD \cdot AQ = AE \cdot AC$ and so the points B, F, E, and C are also concyclic.



Let T be the intersection of BC and FE.

Claim. $TD^2 = TB \cdot TC = TF \cdot TE$.

Proof. We will prove that the circles (DEF) and (BDC) are tangent to each other. Indeed, using the above arguments, we get

$$\angle BDF = \angle AFD - \angle ABD = (180^\circ - \angle FAD - \angle FDA) - (\angle ABC - \angle DBC)$$
$$= 180^\circ - \angle FAD - \angle ABC = 180^\circ - \angle DAE - \angle FEA = \angle FED + \angle ADE = \angle FED + \angle DCB,$$

which implies the desired tangency.

Since the points B, C, E, and F are concyclic, the powers of the point T with respect to the circles (BDC) and (EDF) are equal. So their radical axis, which coincides with the common tangent at D, passes through T, and hence $TD^2 = TE \cdot TF = TB \cdot TC$. □

Solution 1. Let TA intersect the circle (ABC) again at M. Due to the circles $(BCEF)$ and $(AMCB)$, and using the above Claim, we get $TM \cdot TA = TF \cdot TE = TB \cdot TC = TD^2$; in particular, the points A, M, E, and F are concyclic.

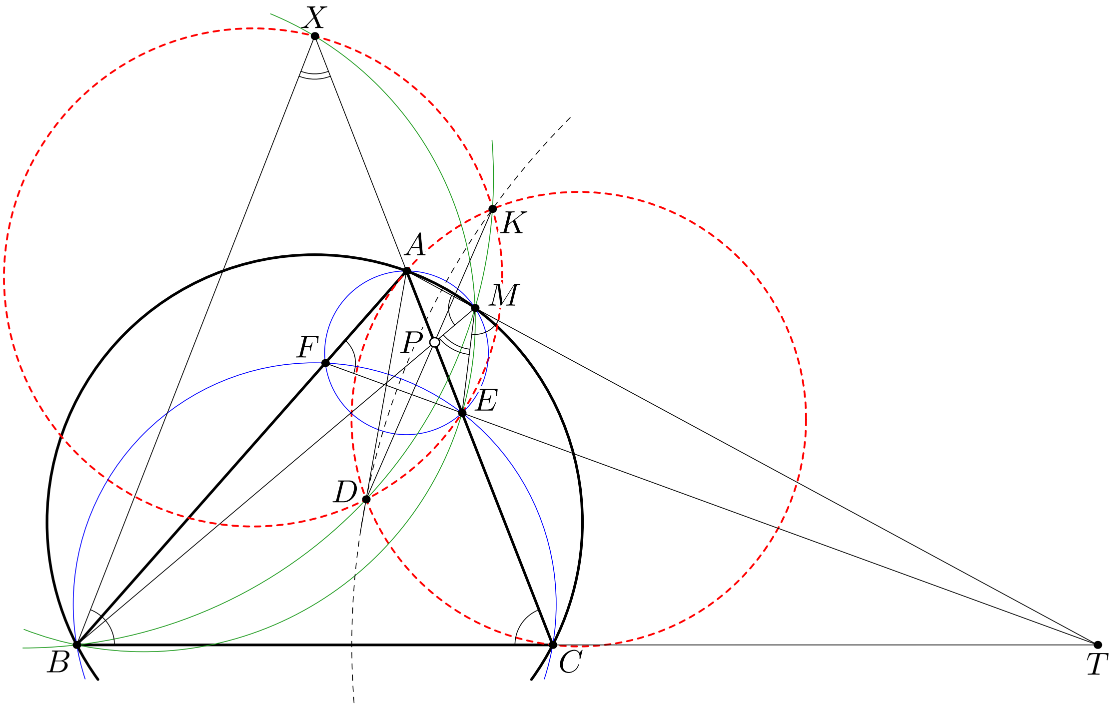
Under the inversion with centre T and radius TD, the point M maps to A, and B maps to C, which implies that the circle (MBD) maps to the circle (ADC). Their common point D lies on the circle of the inversion, so the second intersection point K also lies on that circle, which means $TK = TD$. It follows that the point T and the centres of the circles (KDE) and (ADC) lie on the perpendicular bisector of KD.

Since the center of (ADC) is O_1, it suffices to show now that the points D, K, E, and X are concyclic (the center of the corresponding circle will be O_2).

The lines BM, DK, and AC are the pairwise radical axes of the circles $(ABCM)$, $(ACDK)$ and $(BMDK)$, so they are concurrent at some point P. Also, M lies on the circle (AEF), thus

$$\measuredangle(EX, XB) = \measuredangle(CX, XB) = \measuredangle(XC, BC) + \measuredangle(BC, BX) = 2\measuredangle(AC, CB)$$
$$= \measuredangle(AC, CB) + \measuredangle(EF, FA) = \measuredangle(AM, BM) + \measuredangle(EM, MA) = \measuredangle(EM, BM),$$

so the points M, E, X, and B are concyclic. Therefore, $PE \cdot PX = PM \cdot PB = PK \cdot PD$, so the points E, K, D, and X are concyclic, as desired.



Comment 1. We present here a different solution which uses similar ideas.

Perform the inversion ι with centre T and radius TD. It swaps B with C and E with F; the point D maps to itself. Let $X' = \iota(X)$. Observe that the points E, F, X, and X' are concyclic, as well as the points B, C, X, and X'. Then

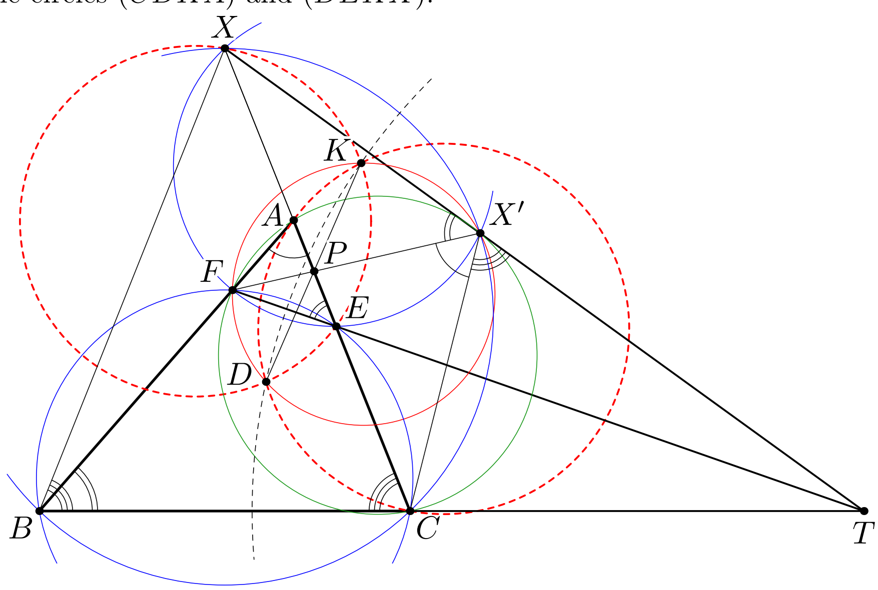
$$\begin{aligned}\measuredangle(CX', X'F) &= \measuredangle(CX', X'X) + \measuredangle(X'X, X'F) = \measuredangle(CB, BX) + \measuredangle(EX, EF)\\ &= \measuredangle(XC, CB) + \measuredangle(EC, EF) = \measuredangle(CA, CB) + \measuredangle(BC, BF) = \measuredangle(CA, AF),\end{aligned}$$

therefore the points C, X', A, and F are concyclic.

Let $X'F$ intersect AC at P, and let K be the second common point of DP and the circle (ACD). Then

$$PK \cdot PD = PA \cdot PC = PX' \cdot PF = PE \cdot PX;$$

hence, the points K, X, D, and E lie on some circle ω_1, while the points K, X', D, and F lie on some circle ω_2. (These circles are distinct since $\angle EXF + \angle EDF < \angle EAF + \angle DCB + \angle DBC < 180^\circ$). The inversion ι swaps ω_1 with ω_2 and fixes their common point D, so it fixes their second common point K. Thus $TD = TK$ and the perpendicular bisector of DK passes through T, as well as through the centres of the circles $(CDKA)$ and $(DEKX)$.



Solution 2. We use only the first part of the Common remarks, namely, the facts that the tuples (C, D, Q, E) and (B, C, E, F) are both concyclic. We also introduce the point $T = BC \cap EF$. Let the circle (CDE) meet BC again at E_1. Since $\angle E_1CQ = \angle DCE$, the arcs DE and QE_1 of the circle (CDQ) are equal, so $DQ \parallel EE_1$.

Since $BFEC$ is cyclic, the line AD forms equal angles with BC and EF, hence so does EE_1. Therefore, the triangle EE_1T is isosceles, $TE = TE_1$, and T lies on the common perpendicular bisector of EE_1 and DQ.

Let U and V be the centres of circles (ADE) and $(CDQE)$, respectively. Then UO_1 is the perpendicular bisector of AD. Moreover, the points U, V, and O_2 belong to the perpendicular bisector of DE. Since $UO_1 \parallel VT$, in order to show that O_1O_2 passes through T, it suffices to show that

$$\frac{O_2U}{O_2V} = \frac{O_1U}{TV}. \tag{1}$$

Denote angles A, B, and C of the triangle ABC by α, β, and γ, respectively. Projecting onto AC we obtain

$$\frac{O_2U}{O_2V} = \frac{(XE - AE)/2}{(XE + EC)/2} = \frac{AX}{CX} = \frac{AX}{BX} = \frac{\sin(\gamma - \beta)}{\sin \alpha} \tag{2}$$

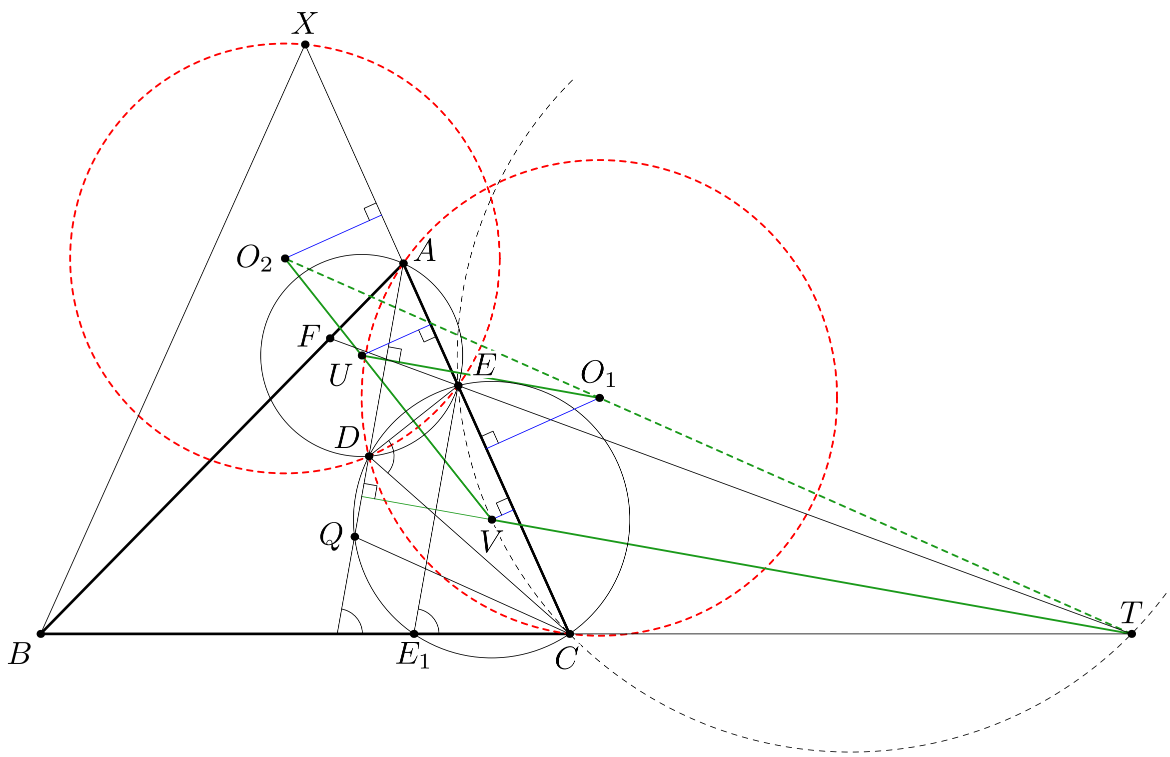
The projection of O_1U onto AC is $(AC - AE)/2 = CE/2$; the angle between O_1U and AC is $90^\circ - \alpha/2$, so

$$\frac{O_1U}{EC} = \frac{1}{2\sin(\alpha/2)} \tag{3}$$

Next, we claim that E, V, C, and T are concyclic. Indeed, the point V lies on the perpendicular bisector of CE, as well as on the internal angle bisector of $\angle CTF$. Therefore, V coincides with the midpoint of the arc CE of the circle (TCE).

Now we have $\angle EVC = 2\angle EE_1C = 180^\circ - (\gamma - \beta)$ and $\angle VET = \angle VE_1T = 90^\circ - \angle E_1EC = 90^\circ - \alpha/2$. Therefore,

$$\frac{EC}{TV} = \frac{\sin \angle ETC}{\sin \angle VET} = \frac{\sin(\gamma - \beta)}{\cos(\alpha/2)}. \tag{4}$$



Recalling (2) and multiplying (3) and (4) we establish (1):

$$\frac{O_2U}{O_2V}=\frac{\sin(\gamma-\beta)}{\sin\alpha}=\frac{1}{2\sin(\alpha/2)}\cdot\frac{\sin(\gamma-\beta)}{\cos(\alpha/2)}=\frac{O_1U}{EC}\cdot\frac{EC}{TV}=\frac{O_1U}{TV}.$$

Solution 3. Notice that $\angle AQE=\angle QCB$ and $\angle AQF=\angle QBC$; so, if we replace the point D with Q in the problem set up, the points E, F, and T remain the same. So, by the Claim, we have $TQ^2=TB\cdot TC=TD^2$.

Thus, there exists a circle Γ centred at T and passing through D and Q. We denote the second meeting point of the circles Γ and (ADC) by K. Let the line AC meet the circle (DEK) again at Y; we intend to prove that $Y=X$. As in Solution 1, this will yield that the point T, as well as the centres O_1 and O_2, all lie on the perpendicular bisector of DK.

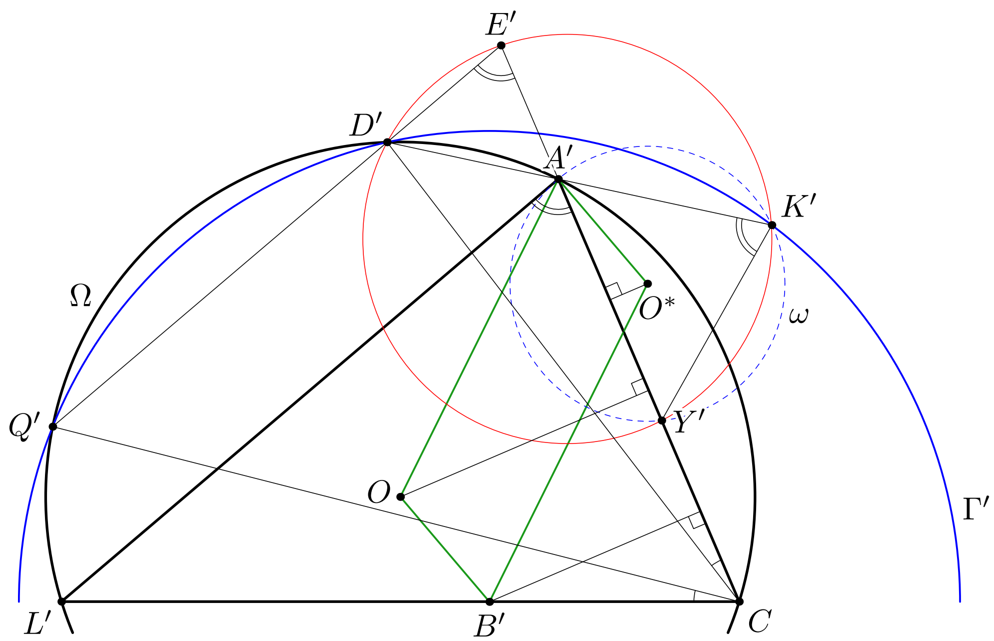
Let $L=AD\cap BC$. We perform an inversion centred at C; the images of the points will be denoted by primes, e.g., A' is the image of A. We obtain the following configuration, constructed in a triangle $A'CL'$.

The points D' and Q' are chosen on the circumcircle Ω of $A'L'C$ such that $\measuredangle(L'C,D'C)=\measuredangle(Q'C,A'C)$, which means that $A'L'\parallel D'Q'$. The lines $D'Q'$ and $A'C$ meet at E'.

A circle Γ' centred on CL' passes through D' and Q'. Notice here that B' lies on the segment CL', and that $\angle A'B'C=\angle BAC=2\angle LAC=2\angle A'L'C$, so that $B'L'=B'A'$, and B' lies on the perpendicular bisector of $A'L'$ (which coincides with that of $D'Q'$). All this means that B' is the centre of Γ'.

Finally, K' is the second meeting point of $A'D'$ and Γ', and Y' is the second meeting point of the circle $(D'K'E')$ and the line $A'E'$, We have $\measuredangle(Y'K',K'A')=\measuredangle(Y'E',E'D')=\measuredangle(Y'A',A'L')$, so $A'L'$ is tangent to the circumcircle ω of the triangle $Y'A'K'$.

Let O and O^* be the centres of Ω and ω, respectively. Then $O^*A'\perp A'L'\perp B'O$. The projections of vectors $\overrightarrow{O^*A'}$ and $\overrightarrow{B'O}$ onto $K'D'$ are equal to $\overrightarrow{K'A'}/2=\overrightarrow{K'D'}/2-\overrightarrow{A'D'}/2$. So $\overrightarrow{O^*A'}=\overrightarrow{B'O}$, or equivalently $\overrightarrow{A'O}=\overrightarrow{O^*B'}$. Projecting this equality onto $A'C$, we see that the projection of $\overrightarrow{O^*B'}$ equals $\overrightarrow{A'C}/2$. Since O^* is projected to the midpoint of $A'Y'$, this yields that B' is projected to the midpoint of CY', i.e., $B'Y'=B'C$ and $\angle B'Y'C=\angle B'CY'$. In the original figure, this rewrites as $\angle CBY=\angle BCY$, so Y lies on the perpendicular bisector of BC, as desired.



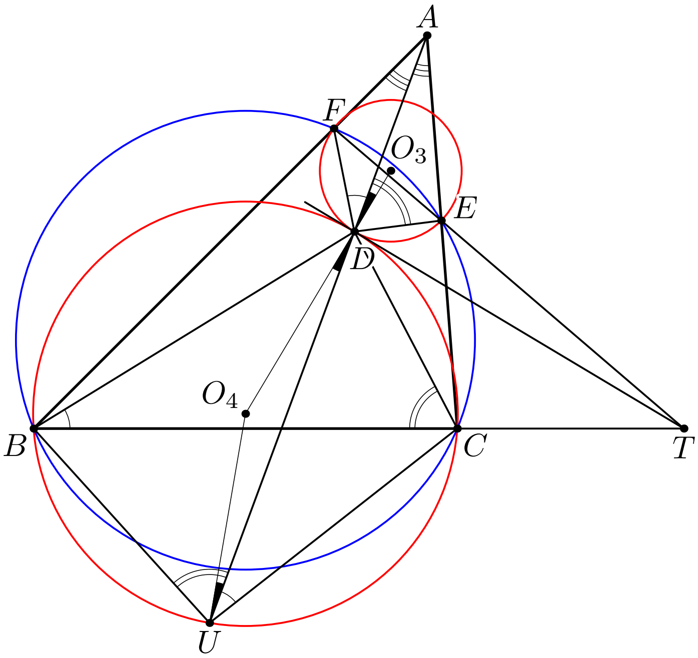
Comment 2. The point K appears to be the same in Solutions 1 and 3 (and Comment 1 as well). One can also show that K lies on the circle passing through A, X, and the midpoint of the arc BAC.

Comment 3. There are different proofs of the facts from the Common remarks, namely, the cyclicity of B, C, E, and F, and the Claim. We present one such alternative proof here.

We perform the composition ϕ of a homothety with centre A and the reflection in AD, which maps E to B. Let $U = \phi(D)$. Then $\measuredangle(BC, CD) = \measuredangle(AD, DE) = \measuredangle(BU, UD)$, so the points B, U, C, and D are concyclic. Therefore, $\measuredangle(CU, UD) = \measuredangle(CB, BD) = \measuredangle(AD, DF)$, so $\phi(F) = C$. Then the coefficient of the homothety is $AC/AF = AB/AE$, and thus points C, E, F, and B are concyclic.

Denote the centres of the circles (EDF) and $(BUCD)$ by O_3 and O_4, respectively. Then $\phi(O_3) = O_4$, hence $\measuredangle(O_3D, DA) = -\measuredangle(O_4U, UA) = \measuredangle(O_4D, DA)$, whence the circle (BDC) is tangent to the circle (EDF).

Now, the radical axes of circles (DEF), (BDC) and $(BCEF)$ intersect at T, and the claim follows.

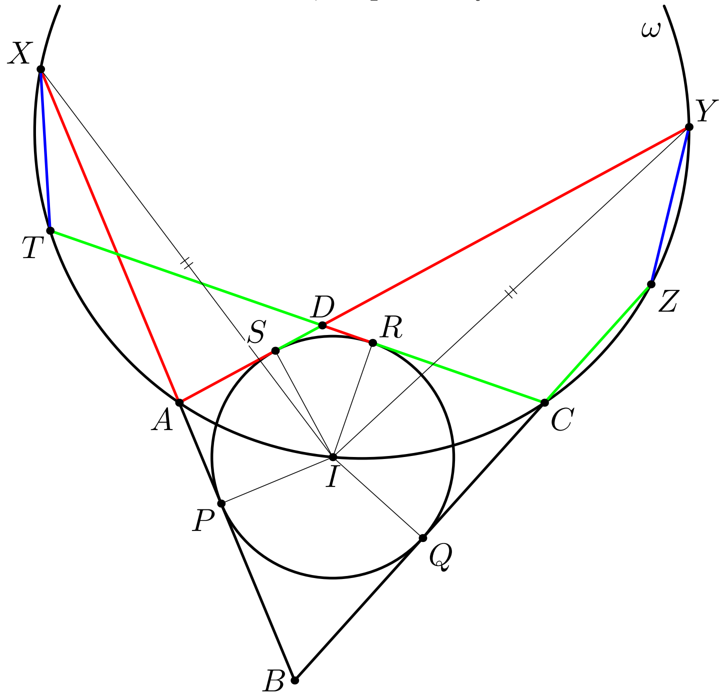


This suffices for Solution 1 to work. However, Solutions 2 and 3 need properties of point Q, established in Common remarks before Solution 1.

Comment 4. In the original problem proposal, the point X was hidden. Instead, a circle γ was constructed such that D and E lie on γ, and its center is collinear with O_1 and T. The problem requested to prove that, in a fixed triangle ABC, independently from the choice of D on the bisector of $\angle BAC$, all circles γ pass through a fixed point.

Problem 4. Let $ABCD$ be a convex quadrilateral circumscribed around a circle with centre I. Let ω be the circumcircle of the triangle ACI. The extensions of BA and BC beyond A and C meet ω at X and Z, respectively. The extensions of AD and CD beyond D meet ω at Y and T, respectively. Prove that the perimeters of the (possibly self-intersecting) quadrilaterals $ADTX$ and $CDYZ$ are equal.

Solution. The point I is the intersection of the external bisector of the angle TCZ with the circumcircle ω of the triangle TCZ, so I is the midpoint of the arc TCZ and $IT = IZ$. Similarly, I is the midpoint of the arc YAX and $IX = IY$. Let O be the centre of ω. Then X and T are the reflections of Y and Z in IO, respectively. So $XT = YZ$.



Let the incircle of $ABCD$ touch AB, BC, CD, and DA at points P, Q, R, and S, respectively.

The right triangles IXP and IYS are congruent, since $IP = IS$ and $IX = IY$. Similarly, the right triangles IRT and IQZ are congruent. Therefore, $XP = YS$ and $RT = QZ$.

Denote the perimeters of $ADTX$ and $CDYZ$ by P_{ADTX} and P_{CDYZ} respectively. Since $AS = AP$, $CQ = RC$, and $SD = DR$, we obtain

$$P_{ADTX} = XT + XA + AS + SD + DT = XT + XP + RT$$
$$= YZ + YS + QZ = YZ + YD + DR + RC + CZ = P_{CDYZ},$$

as required.

Comment 1. After proving that X and T are the reflections of Y and Z in IO, respectively, one can finish the solution as follows. Since $XT = YZ$, the problem statement is equivalent to

$$XA + AD + DT = YD + DC + CZ. \tag{1}$$

Since $ABCD$ is circumscribed, $AB - AD = BC - CD$. Adding this to (1), we come to an equivalent equality $XA + AB + DT = YD + BC + CZ$, or

$$XB + DT = YD + BZ. \tag{2}$$

Let $\lambda = \frac{XZ}{AC} = \frac{TY}{AC}$. Since $XACZ$ is cyclic, the triangles ZBX and ABC are similar, hence

$$\frac{XB}{BC} = \frac{BZ}{AB} = \frac{XZ}{AC} = \lambda.$$

It follows that $XB = \lambda BC$ and $BZ = \lambda AB$. Likewise, the triangles TDY and ADC are similar, hence

$$\frac{DT}{AD} = \frac{DY}{CD} = \frac{TY}{AC} = \lambda.$$

Therefore, (2) rewrites as $\lambda BC + \lambda AD = \lambda CD + \lambda AB$.

This is equivalent to $BC + AD = CD + AB$ which is true as $ABCD$ is circumscribed.

Comment 2. Here is a more difficult modification of the original problem, found by the PSC.

Let $ABCD$ be a convex quadrilateral circumscribed around a circle with centre I. Let ω be the circumcircle of the triangle ACI. The extensions of BA and BC beyond A and C meet ω at X and Z, respectively. The extensions of AD and CD beyond D meet ω at Y and T, respectively. Let $U = BC \cap AD$ and $V = BA \cap CD$. Let I_U be the incentre of UYZ and let J_V be the V-excentre of VXT. Then $I_U J_V \perp BD$.

Problem 5. Two squirrels, Twinkelberry and Nutkin, have collected 2021 walnuts for the winter. Twinkelberry numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circle in the ground around their favourite tree. The next morning Twinkelberry notices Nutkin had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Twinkelberry decides to reorder the walnuts by performing a sequence of 2021 moves. In the k-th move, Twinkelberry swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, in the k-th move, Twinkelberry swaps some walnuts a and b such that $a < k < b$.

Common remarks. In all solutions below, we use the following notation.

Before the k-th move, each walnut a with $a < k$ is called *used*, and every walnut b with $b \geq k$ is called *active*. Thus, at the beginning of the process, all walnuts are active, and on each move exactly one walnut changes its state from being active to being used.

We usually argue indirectly (without claiming that explicitly). Under an indirect assumption, on the k-th move, the squirrel swaps either two walnuts $a, b < k$, or two walnuts $a, b > k$. In the former case we say that walnut k (and the number k) are *large*, otherwise they are *small*. Clearly, 1 is small, while 2021 is large.

Solution 1a. At any moment in the process, the used walnuts are split into several groups consisting of one or more contiguous used walnuts each; different groups are separated by active walnuts.

We prove by induction on $1 \leq k \leq 2020$ that, after the k-th move, all groups of used walnuts have odd sizes. The base case $k = 1$ is obvious. To prove the step, consider the current, k-th, move. Two cases are possible:

Case 1: k is small.

In this case, both neighbours of walnut k remain active, so k forms a separate group.

Case 2: k is large.

In this case, the neighbours of k are both used, so they belong to two groups containing, say, p and q walnuts, respectively (both p and q are odd). Now, when k becomes used, those two groups merge into a single group consisting of an odd number of walnuts — namely, $p + q + 1$.

Now, after the 2020-th move, the 2020 used walnuts should form several groups of odd sizes. However, they in fact form just one group of size 2020. This is a contradiction.

Solution 1b. We act similarly to Solution 1a. At any moment in the process, the *active* walnuts are split into several groups consisting of one or more contiguous active walnuts each; different groups are separated by used walnuts.

Now the claim is that, after the k-th move, there exists a (nonempty) group of active walnuts of an even size. The base case $k = 1$ is trivial again. For the step, notice that walnut k is active before the k-th move, so it belongs to some group of active walnuts. We distinguish the following cases.

Case 1: Walnut k is a boundary walnut in some group.

If the k-th move is not prohibited, this means that walnut k forms a separate group, so its size is odd. The other groups remain the same, so an even-sized group persists.

Case 2: Walnut k is not a boundary walnut in its group.

In this case, this group is split into two groups. If the size of the group was odd, then an even-sized group persists. Otherwise, one of the two obtained groups has a (nonzero) even size. The step is proved.

Now, after the 2020-th move, there is only one active group consisting of one walnut. This contradicts the above claim.

Comment. Solution 1b can be obtained from solution 1a by the *Time Reversal argument*: one can reverse the time and relabel the walnuts as $2021, 2020, \ldots, 1$; then a valid process turns into another valid process.

Solution 1′. Notice that, at move k, the squirrel swaps either two used walnuts (if k is large), or two active ones (if k is small). Therefore, if a position contains a used walnut at some moment, it will contain such walnut ever since.

Therefore, each position is central for at most one move — and hence exactly one.

Consider now two adjacent positions, p and q, which were central at moves a and b, respectively ($a < b$). Assume that both a and b are small. Then, at move b, the position p is occupied by a used walnut, so its number is less than b. This cannot happen since b is small.

Similarly, it cannot happen that both a and b are large. Therefore, the positions where the large and small moves are performed alternate, which is impossible since 2021 is odd.

Solution 2. We aim at proving that, for every k,

(i) if k is small, then two large walnuts are swapped on the k-th move; and, similarly,

(ii) if k is large, then two small walnuts are swapped on the k-th move.

Claim 1. Assume that, after move t, walnut x has a used neighbour. Then x will have a used neighbour after every subsequent move. In particular, if $t \leq x$, then x is large.

Proof. Let a be a used neighbour of x before the y-th move. If neither of x and a is moved on that move, they remain adjacent. If x is moved on the y-th move, then, after the move, y is its used neighbour. If a is moved at the move, then a is swapped with some walnut b, and both a and b should be smaller than y (since a is used); so b is a new used neighbour of x after the move. □

Now, if k is small, then after the k-th move the two swapped walnuts are still active and have a used neighbour. Therefore, they are both large. This establishes (i).

Claim 2. Assume that walnut k is large. Then, after the k-th move, both its neighbours will be always used.

Proof. Immediately after the k-th move, both neighbours of walnut k have numbers smaller than k, so they are both used. If any of them is swapped with some other walnut, then the new walnut is also used, by the argument similar to that in Claim 1. Thus, x does not get an active neighbour, so x cannot move anymore. □

Hence, a large used walnut never moves.

Now we are prepared to prove (ii). Aiming at a contradiction, suppose now that walnut a is large, and, at the a-th move, walnuts b and c were swapped, and b is large. Since $a > b$, walnut b was used earlier, so at the a-th move both its neighbours must be used, which is false for a. This contradiction finishes the solution.

Thus (i) and (ii) are proved. It follows that, at any move two swapped walnuts are of the same type (large or small). So, the type of a walnut on an arbitrary position is preserved during the process. Moreover, those types alternate; this contradicts the parity of 2021.

Comment. The proof of (ii) may be also obtained from that of (i) (or vice versa) by the Time Reversal argument.

Solution 2′. Imagine that we write down the numbers $1, 2, \ldots, 2021$ in a row. Some of them are small, the others are large; so the numbers are split into groups of contiguous numbers of the same type. To formalise this, introduce the numbers $0 = \ell_0 < s_1 < \ell_1 < s_2 < \cdots < s_m < \ell_m = 2021$ so that:

- the numbers $\ell_{i-1}+1, \ell_{i-1}+2, \ldots, s_i$ are small, for all $i = 1, 2, \ldots, m$; and
- the numbers $s_i + 1, s_i + 2, \ldots, \ell_i$ are large, for all $i = 1, 2, \ldots, m$:

$$1, 2, \ldots, s_1, s_1+1, \ldots, \ell_1, \ell_1+1, \ldots, s_2, s_2+1, \ldots, \ell_{k-1}+1, \ldots, s_k, s_k+1, \ldots, \ell_k, \ldots$$

(colours indicate small and large numbers)

We first reduce the problem to the following claim, and then we prove the claim.

Claim. For every $1 \le k \le m$, after the ℓ_k-th move, we have the following:

1°) no two used small numbers are adjacent; and

2°) each used large number is adjacent to two used small numbers.

The problem statement follows from the Claim for $k = m$, since it follows that, after the last moves, large and small numbers alternate in the circle; this is impossible since 2021 is odd.

Proof of the Claim. Arguing indirectly, choose the smallest k violating the claim. Consider the situation after the ℓ_{k-1}-th move; that situation satisfies both 1° and 2° (if $k = 1$, both statements are void).

Set $\ell := \ell_{k-1}$ and $s := s_k$. Let us call a number *important* if it does not exceed ℓ_k. Clearly, any small important number does not exceed s. The forthcoming moves $\ell + 1, \dots, s$ are small, and the next moves $s + 1, \dots, \ell_k$ are large; we will show that, after all those moves, both 1° and 2° hold, which is a contradiction.

Suppose first that, between the ℓ-th and the $(s + 1)$-th move, there was a moment when two small important numbers were adjacent. If such a pair existed immediately after the ℓ-th move, then one of its numbers should be between $\ell + 1$ and s (if both are less or equal than ℓ, then they are used and 1° fails); then there should exist a move b (with $\ell < b \le s$) on which some small important number a was moved (clearly, in this case we have $a > b$, since b is small). Otherwise, if an adjacent pair appeared on some move $b > \ell$, then one element a of that pair was moved on the b-th move.

So, in any case, we proved the existence of a pair of $a > b$ such that

- $\ell < b < a \le s$; and
- a was moved on the b-th move.

Call such a pair *bad*. Choose a bad pair $a > b$ with the smallest value of a; among those, we choose the pair with the largest value of b. Consider the process between the b-th and the a-th moves. By the choice of a, no small walnut $a' < a$ was moved during that period. On the other hand, a and b were not adjacent on the a-th move, since a is small. Therefore, at some move c with $b < c < a$, walnut a moved away from b, which could happen only if a and c were adjacent. This contradicts the choice of b.

Hence, no two important small walnuts were adjacent during the moves $\ell + 1, \dots, s$. Hence they preserve their positions during those moves. On the other hand, we know that, just after the ℓ-th move, all used large walnuts ($\le \ell$) were surrounded by used small walnuts, so they are "protected" by those small walnuts from being moved. This protection is preserved during moves $\ell + 1, \dots, s$, so the used large walnuts are still protected by used small ones.

Now we look at the important large numbers $s + 1, \dots, \ell_k$. At the $(s + 1)$-th move, two used numbers x and y should be swapped. Since all used large numbers are protected, both x and y are small (and used); so walnut $s + 1$ is protected by x and y just after the $(s + 1)$-th move.

Repeating the arguments verbatim for $s + 2, s + 3, \dots, \ell_k$, we obtain that 2° holds after the ℓ_k-th move. But this means that, on each of those moves, two small used numbers were swapped, so no two such numbers could become adjacent. So, after the ℓ_k-th move, 1° also holds. ☐

Solution 3. After each move, we consider all triples of (clockwise) contiguous positions. Say that a triple containing walnuts a, b, c (in this order) is *monotone* if either $a > b > c$ or $a < b < c$ holds. Say that a monotone triple (a, b, c) is *fine* if b is active.

Lemma 1. At any moment, the number of the monotone triples is odd.

Proof. It suffices to prove that the number of non-monotone triples is even. Such triples correspond to local maxima and local minima of the cyclic sequence; since those alternate, the lemma is proved. ☐

Arguing indirectly, we assume that none of the moves swaps the boundary elements of a monotone triple. Under this assumption, we prove the following lemma.

Lemma 2. For every k, there is an odd number of fine triples after the k-th move.

Notice here that Lemma 2 immediately gives a contradiction, since there are no fine triples after the 2021-th move.

Proof of Lemma 2. Induction on k. The base case $k = 0$ holds due to Lemma 1.

To perform the step, assume that there is an odd number of fine triples after the $(k-1)$-th move. We say that three positions in the circle form a *special triple* if that triple either became fine (while it was not such before), or stopped being fine on the k-th move.

Consider an arc of the circle before the k-th move; assume that it contains walnuts

$$\dots, a, b, c, k, d, e, f, \dots$$

in this order. At the moment, the triple (c, k, d), that is replaced with (d, k, c), is not fine by our assumption (neither it became such). So, special triples could appear only at the positions occupied by (a, b, c), (b, c, k), (k, d, e), or (d, e, f). We show that the first two triples are either both special, or both non-special. The same holds for the other two triples; this will finish the step of induction.

We distinguish two cases.

Case 1: k is large, so that $c, d < k$.

Walnuts c and d were used before the k-th move; therefore, (b, c, k) is not special. Moreover, if the triple (a, b, c) is special, then it went from monotone to not monotone or vice versa while changing to (a, b, d), so the number b lies between c and d (meaning that $\min\{c, d\} < b < \max\{c, d\}$), so number b was also used before the k-th move. But this means that (a, b, c) is not special. Thus, in this case neither of the two triples is special.

Case 2: k is small, so that $c, d > k$, and walnuts c and d are both active.

If the triple (a, b, c) is special, then b should lie between the numbers c and d. If the triple (b, c, k) is special, then since c and d are both greater than k, the number b again should lie between the numbers c and d. If any condition holds, b is active and so exactly one of the triples (a, b, c) and (a, b, d) is monotone, as well as exactly one of (b, c, k) and (b, d, k). Hence if one of the triples is special, both triples are special. □

Solution 3′. Let us say that a pair of two adjacent walnuts is *active* if at least one of its elements is active, otherwise we say that this pair is *used*.

Denote by S_k the number of active pairs after the k-th move. Assume, to the contrary, that on each move the squirrel swaps either two used walnuts or two active ones. Let us prove that after each swap the parity of S_k is preserved. If the k-th walnut has two active neighbours, then all active pairs are preserved, and $S_k = S_{k-1}$. If the k-th walnut has two used neighbours, then $S_k = S_{k-1} - 2$, as the two pairs containing k-th walnut become used. In both cases, the parity of S_k is preserved. This contradicts the fact that $S_0 = 2021$ and $S_{2021} = 0$.

Solution 4. Say that the k-th move is *legal* if, at the move, either two walnuts with numbers less than k or two walnuts with numbers greater than k are swapped. We assume that all moves in the process are legal.

Call a usual legal move a *swap*. We now introduce a different type of moves: namely, we allow, at the k-th move, to check if the neighbours of walnut k either both have numbers less than k or both have numbers smaller than k. If this property is satisfied, we perform nothing (such legal move will be referred to as an *inspection*); otherwise the move is not legal.

Now the problem follows from the following, more general claim.

Claim. In any arrangement of 2021 walnuts, one cannot do 2021 legal moves where the k-th move is either a swap or an inspection.

Proof. We run an induction on the number of swaps in the sequence of moves. We will replace swaps with inspections one by one.

In the base case, where there are no swaps, the walnuts never move, but all the inspections should be performed. So every walnut's number is either less than those of both its neighbours or larger than those of both its neighbours. So local maxima and local minima alternate, which is impossible since 2021 is odd.

To perform the step, consider the first swap happening, e.g., at the k-th move. We distinguish two cases.

Case 1: Walnut k is large.

We consider the same initial arrangement and apply the same moves, replacing the swap on the k-th move by the inspection. On each subsequent move, if the number a or b is compared with the current move's number, it is less than that number; so replacing a with b does not break legality of the moves.

Case 2: Walnut k is small.

Before move k, we have done only inspections; so we can consider the alternative initial arrangement, where all the walnuts are at their places except for walnuts a and b which are swapped. Now we can run the process from the changed initial configuration, with all the same moves, except move k being an inspection. All the moves remain legal, for the same argument as above.

Thus, in both cases, we decreased the number of swaps; it remains to apply the induction hypothesis. □

Problem 6. Let $m \geqslant 2$ be an integer, A be a finite set of (not necessarily positive) integers, and $B_1, B_2, B_3, \ldots, B_m$ be subsets of A. Assume that for each $1 \leqslant k \leqslant m$ the sum of the elements of B_k is m^k. Prove that A contains at least $m/2$ elements.

Solution. Let $A = \{a_1, \ldots, a_k\}$. Assume that, on the contrary, $k = |A| < m/2$. Let

$$s_i := \sum_{j: a_j \in B_i} a_j$$

be the sum of elements of B_i. We are given that $s_i = m^i$ for $i = 1, \ldots, m$.

Now consider all m^m expressions of the form

$$f(c_1, \ldots, c_m) := c_1 s_1 + c_2 s_2 + \ldots + c_m s_m, \ c_i \in \{0, 1, \ldots, m-1\} \text{ for all } i = 1, 2, \ldots, m.$$

Note that every number $f(c_1, \ldots, c_m)$ has the form

$$\alpha_1 a_1 + \ldots + \alpha_k a_k, \ \alpha_i \in \{0, 1, \ldots, m(m-1)\}.$$

Hence, there are at most $(m(m-1)+1)^k < m^{2k} < m^m$ distinct values of our expressions; therefore, at least two of them coincide.

Since $s_i = m^i$, this contradicts the uniqueness of representation of positive integers in the base-m system.

Comment 1. For other rapidly increasing sequences of sums of B_i's the similar argument also provides lower estimates on $k = |A|$. For example, if the sums of B_i are equal to $1!$, $2!$, $3!$, $\ldots$, $m!$, then for any fixed $\varepsilon > 0$ and large enough m we get $k \geqslant (1/2 - \varepsilon)m$. The proof uses the fact that the combinations $\sum c_i i!$ with $c_i \in \{0, 1, \ldots, i\}$ are all distinct.

Comment 2. The problem statement holds also if A is a set of real numbers (not necessarily integers), the above proofs work in the real case.